The Effects of Using Representations in Elementary Mathematics: Meta-Analysis of Research

Andrzej Sokolowski
Lone Star College, Houston, Texas, USA

Abstract

The current study provides a meta-analysis of global research on using representations to support the learning of mathematics in Pre-K through Grade 5. A total of 13 primary studies encompassing 1,941 subjects was analyzed. The weighted mean effect size for the 13 studies was reported to be ES = 0.53 (SE = 0.05). A 95% confidence interval around the overall mean – C_{lower} = 0.42 and C_{upper} = 0.63 – proved its statistical significance and its relative precision. The calculated effect size signifies strong, robust support for the use of representations in Pre-K through Grade 5 mathematics classes and highlights the importance of providing students with opportunities to construct and explore transitioning between various forms of representations. Moderator analysis revealed differences among the effects due to a different type of representation, grade levels, and concepts taught. A synthesis of moderator effects allowed for a formulation of a general way of applying representations that produces maximum learning effects and that the teachers can adopt in their school practice. While the effect sizes provided a means of determining the most effective ways of applying representations, questions about how to develop students' transitioning from one representation to another remain unsolved. A further discussion of the impact of the study findings beyond the boundaries of elementary mathematics classrooms follows.

Keywords: meta-analysis, mathematical representations, internal and external representations, mathematics teaching and learning, elementary school

Introduction

It is well documented that reading and understanding a mathematical context requires embodying various abstract entities such as symbols, graphs, and tasks encoded symbolically in a language that students can comprehend. Given that elementary students are at the stage of developing their abstract thinking, designing effective teaching strategies that would allow such communication is not an easy task for mathematics educators and curriculum designers thus attempts are made to make the process more accessible for the learners. Research findings (e.g., Clark & Mayer, 2011) show that learning mathematical objects and the development of corresponding mental images are linked. On the other hand, imagination and the ability to construct, retrieve and explore internal representations form foundations for learning of mathematics (Lingefjärd & Ghosh, 2016). The ultimate question that was posited in this study was what representations are the most accessible to an elementary math student.

Researchers (e.g., Hoffler & Leutner, 2007) have determined that people learn more deeply from words supported by graphics than from words alone. This finding corresponds with the modern view on mathematical learning, which claims that utilizing multiple representations and making connections between graphical, symbolic, and verbal descriptions of mathematical relationships will empower and simultaneously help students develop a deeper understanding of mathematical relationships and concepts (National Council of Teachers of Mathematics [NCTM], 2000; Porzio, 1999). Following this notion, a general agreement exists that using different mathematical forms of representations and translating between these forms, are key skills in mathematics (e.g., Ainsworth, Bibby, & Wood, 2002). In order to respond effectively to learners' perception, further research is necessary. Nistal, Van Dooren, & Verschaffel (2012) noted that there was a need for research that would focus especially on the contextual factors that promote flexible representation choice for students in mathematics. It was hoped that this study would shed more light into this area.

Representations, especially their graphical forms, can also be perceived as learning experiences that are transmitted to the learner by pictorial media (Clark & Mayer, 2011). As such, they help the learner identify meaningful pieces of information and link the information with the learner's prior experiences. Although the constructs of using diverse forms of representations to enhance the development of mathematical concepts and problem-solving techniques have been widely researched (e.g., Jitendra, Star, Rodriguez, Lindell, & Someki, 2011; Weber-Russell & LeBlanc, 2004), a formal meta-analysis in this domain was not found using standard library search engines. Students' early experiences with the content of mathematics might have a profound impact on their further engagement and success in the subject. Therefore this study emerged to fill the gap and to provide a broader view of using representations to support the learning of mathematics in elementary school.

Theoretical Background

Representations and Constructivist Learning Theory

The effect of using representations is not new to the mathematics education community. However, it has recently attracted more attention by being supported by the constructivist learning theory that leads contemporary research in education (Cuoco, 2001). By treating mathematical concepts as objects, and thus embodying them with representations that can be observed and manipulated, a construction of mental pictures in the students' minds can be evoked (Dubinski, 1991). Such constructed mental pictures are stored in students' long-term memory and are available for retrieval. Research (Zazkis & Liljedahl, 2004) suggests that one of the ways to induce the process of converting concepts to objects is to create representations and act on them. Clark and Mayer (2011) suggested that knowledge acquisition is based on the following principles of learning: dual channel – people have separate channels for processing visual/pictorial material and auditory/verbal material; limited capacity – people can actively process only a few pieces of information in each channel at one time; and active processing – learning occurs when people engage in appropriate cognitive processing such as attending to relevant material and organizing the material into a coherent structure. Active learning appears as a method that supports the linkage of external representations with internal images.

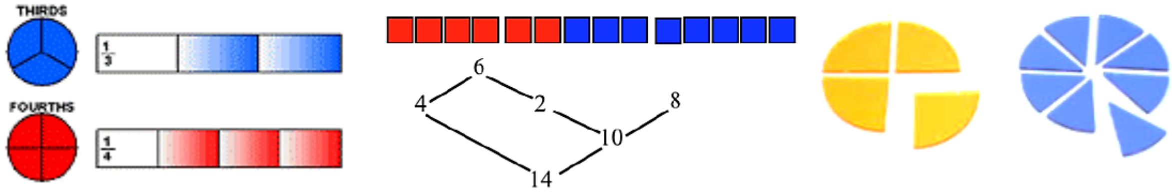
Human cognitive architecture (Paas, Renkl, & Sweller, 2003), states that the most crucial structures affecting the rate of information processing are working memory and long-term memory. Human working memory has limited capacity as opposed to long-term memory, where capacity is unlimited (Kintch, 1998). For the information to be stored in a learner's long-term memory, it needs to be processed initially through its working stage. Being presented with complex information, the learner might feel overwhelmed, which might result in the information not being fully processed. This state will consequently block the information from reaching the learner's long-term memory and from being learned and accumulated. The primary goals of using representations are to reduce the contextual load by converting the information to a visual form and to transmit such information to the learner's visual channel. This process in return will reduce the need for high capacity of working memory. The virtue of using representations is rooted in their capacity to present the knowledge of conveyable graphical embodiments supported by verbal elaborations rather than vice versa. Such knowledge presentation creates appealing conditions for being longer retained and accessible for future usage.

Representations in Mathematics

Representations are broadly defined as passive entities. By learner's active engagement, they are transformed into active semiotic resources (Thomas, Yoon, & Dreyfuss, 2009) and can be stored in a learner's long-term memory. Knowledge externalized by graphics is easily retrievable for analysis and can be readily exhibited and communicated (Ozgun-Koca, 1998). Representations as a means by which individuals make sense of situations (Kaput & Roschelle, 1997) can be expressed in forms of combinations of written information on paper, physical objects, or a carefully constructed arrangement of thoughts. Schnotz (2002) emphasizes the distinction between descriptive (symbolic) and depictive (iconic) representations. While depictive representations are most useful to provide concrete information and are often effective as specific information, descriptive representations usually express abstract information. Duval (2006) claimed that using various representations in mathematics classes is a necessity because only multiple external representations allow learners to utilize the different advantages each representation offers. Falcade, Laborde, & Mariotti, (2007) claimed that the link between external representations and internal representations goes beyond pure analogy in their functioning and rests on the real tie that can be recognized between particular tools (external representations) and particular signs (internal representations).

Each representation of a mathematical object brings some aspects to the fore, simultaneously hiding other aspects of the object and thus affecting the way the object is seen (Laborde, 2007). Representations can also be used to explore aspects of a context that might otherwise not be apparent to a learner; they amplify properties of mathematical structures not easily imaginable (Monk, 2003). In the process of knowledge accumulation, representations are converted into internal images. Mediated by the level of entry into learners' memory systems, representations are categorized as external or internal. Both types of representations are interrelated in the sense that the meaning of an internal representation stored in a learner's long-term memory strongly depends on the learner's perception of its external counterpart.

External representations (see Figure 1) encompass physically embodied, observable configurations – such as pictures, concrete materials, tables, equations, diagrams, and drawings of one-, two-, or three-dimensional figures or various forms of schemata (Jitendra et al., 1998).



A. Pictorial B. Schemata C. Concrete

Figure 1. Examples of representations using in elementary school.
(Source: https://www.google.com)

All these embodiments can be provided in the form of drawings or can be digitalized by computer programs. They can also be generated by the instructor as he/she introduces the representations to the learners. External representations also encompass dynamic graphics, which are generated with the help of technology, for example, graphing calculators or computer-based simulations (Goldin & Shteingold, 2001). Ainsworth and Van Labeke (2004) categorize external representations as time-persistent representations, time-implicit representations, and static representations. In mathematical terms, time-persistent representations are embodied by algebraic functions, time- implicit by relations and static representations would encompass any drawings that students produce, not necessarily placing their products in a coordinate system.

Eisenberg and Dreyfus (1991) noted that students might end up with an incorrect solution if their algebraic skills are not strong; however, if they possess the skills to graphically solve the problem or support its solution process, the graphed representation might serve as a backup or a way of solution verification. Being exposed to mathematical representations, learners "acquire a set of tools that significantly expand their capacity to model and interpret physical, social, and mathematical phenomena" (NCTM, 2000, p. 4). In this regard, external representations can also serve as a means to overcome students' misconceptions in science classes (Thompson & Logue, 2006).

Internal representations encompass mental images corresponding to internal formulations of what human beings perceive through their senses and as such they cannot be directly observed. Internal representations are defined as the knowledge stored in a learner's long-term memory. Internal representations are formulated based on one's interaction with the environment (external representations) and are altered throughout a lifespan. In the process of learning, external representations prompt the emergence of internal representations. Being able to formulate concepts' internal representations through the process of understanding their external embodiments and retrieving the mental pictures plays an essential role in communicating messages in mathematics. Hiebert and Carpenter (1992) maintained that learners establish a strong relationship between created external and internal representations and that the strength of linking these representations determines students ‘understanding.

Furthermore, internal representations of the knowledge accumulated through experiencing visual representations produce stronger impulses in learners’ long-term memory. Enabling these experiences by engaging and intellectually stimulating learners through carefully designed learning environments supported by representations deems to be a significant factor in nurturing effective learning and developing students’ mathematical knowledge. Nitsch et al. (2015) found that that to understand the concept of function, that is central in mathematics curriculum, it important not only to know the different mathematical representations of functional dependency, but also the translations between these forms of representations. For students to develop a holistic understanding of the concept of mathematical functions, they have to be able to identify the connecting elements and to combine these representations.

Synthesis of Prior Research

As the constructivist theory strongly supports the use of representations in the learning process, several research studies have explored the effects of using representations on students’ math concept understanding. These results converge with contemporary theories of cognitive load and multimedia learning principles developed by Clark and Mayer (2011) and have practical implications for mathematical instructional designs. A meta-analysis of 35 independent experimental studies conducted by Haas (2005) shed light on using representations as a means of supporting teaching methods at the secondary school level. Haas concluded that math instruction, supported by multiple representations, manipulatives, and models, produced a high (ES = 0.75) effect size. Schemas, which are defined as generalized representations that link two or more concepts are frequently researched at the Pre-K through fifth-grade level. For example, Jitendra and colleagues (1998) found that having students of Grades 2-6 categorize problems and then having them solve the problems by using schemas produced a positive medium-size learning effect (ES = 0.45). The virtue of using representations embodied by schemas is that they are easily converted by learners into internal representations, and, as such, they can be stored in long-term memory and allow for treating diverse elements of information regarding larger, more general units (Kalyuga, 2006). According to Pape and Tchoshanov (2001), schematic representations also lead to enhanced student problem-solving performance.

Another group of researchers investigated whether representations should be provided to students or if the students should be the producers of representations (e.g., De Bock, Verschaffel, Janssens, Van Dooren, & Claes, 2003; Rosenshine, Meister, & Chapman, 1996). These scholars concluded that if representations are provided, their forms must be sufficiently informative and detailed to be transferrable by students into mathematical algorithms. They also emphasized that having students construct their representations benefits the learners the most. The importance of possessing the ability to transfer a given context (e.g., a story problem) into a representation was highlighted by Jonassen (2003), who claimed that successful problem solving requires the comprehension of relevant textual information along with the capacity to visualize that data and transfer it into a conceptual model. Following Riley, Greeno and Heller (1984), developing students’ abilities to identify the matching representation that helps with problem conceptual understanding should emerge as a priority of elementary mathematics teaching.

Representations are also used to support the introduction of new mathematical concepts. For example, several studies (Tzur, 1999) were conducted on the development of students' notations of fractional parts of areas, called *fair sharing*, which provided a meaningful representation of dividing a whole into parts that were then easily comprehended by elementary students. Hiebert (1988) noted that students' understanding of new ideas strongly depends on the degree to which the learners are engaged in investigating the relations between new representations and the representations whose understanding is already mastered. A study conducted by Ross and Willson (2012) not only supported the claim that mathematics students learn more effectively when instruction focuses on using representations but, moreover, proved that the most effective strategies for building representations are those rooted in constructivist learning theory. The range of using representations in Pre-K through fifth grade is wide, thus synthesizing the experimental research findings and identifying the most effective strategies manifests as a worthy undertaking.

Challenges of Inducing Representations in Pre-K through Grade 5

Investigating the effect of using representations has recently attracted more attention due to being supported by the constructivist learning theory that leads contemporary research in education (Cuoco, 2001). Such constructed mental pictures are stored in students' long-term memory and are available for retrieval. Research (Zazkis & Liljedahl, 2004) suggests that one of the ways to induce the process of converting concepts to objects is to create representations and act on them. Sfard (1991) concluded that the process of transferring abstract mathematical concepts into their mental images is challenging for both the learner and the instructor, who is to guide the learner through the transferring processes. What are the challenges faced by elementary school children as they attempt to embody mathematical structures into visual representations?

Equations and their conceptualization are frequently investigated in K-5 mathematics research. Swafford and Langrall (2000) noted that students generally can make use of various representations and can identify patterns between isolated variables, but they cannot find consistency among a larger set of variables or generalize the patterns and convert them into mathematical forms. Dreyfus (1991) suggested four learning phases with representations: using one representation; (using more than one representation; (making links between parallel representations; and integrating the representations. Representations at the elementary school level encompass general structures used in mathematics such as ratio, rate, percent or newly developed schemata for problem-solving, thus pinpointing and understanding how to uncover these principles acts as a catalyst for selecting correct representation. According to Swafford and Langrall (2000), the emphasis in the curriculum at the pre-algebra level should be on developing and linking multiple representations to generalize problem situations. They concluded that the lack of generalization skills is rooted in instruction focused on reaching only the initial stages of problem analysis and leaving the process of generalization for the students to formulate. A similar conclusion was reached by Deliyianni, Monoyiou, Elia, Georgiou, and Zannettou (2009), who observed that first-graders restricted themselves to providing unique solutions even though the questions required a general pattern formulation. Other researchers (e.g., Lesh & Harel, 2003) have shown that elementary school children bring potent intuitions and sense-making tools, yet how to mediate these intuitions with abstract math concepts to embody these concepts into representations is a challenge still facing the mathematics research community.

Research Methods

Meta-analysis, with its quantitative methods, was used to compute the research findings. Meta-analytic techniques provide tools to assess the learning effect size of treatments, considering a gathered pool of studies as a set of data collected within prescribed criteria. By allowing the measurement of the effect sizes according to the population of participants in primary studies, such undertaking allows for analyzing a larger number of studies that can vary by population sizes and also by the conduct (Gijbels, 2005). Furthermore, meta-analysis allows also for employing subgroup moderator analysis and extracting factors that contribute to the magnitude and direction of the mean effect size.

Research Problems

Based on the prior research, a hypothesis suggesting that using representations in mathematics helps students comprehend abstract mathematical concepts and enhances the skills of the concepts' applications emerged for this study. Understanding the degree to which representations help learners comprehend the different mathematics entities, compared to traditional methods of instruction, constituted the main objective of this study and guided the research questions:

1. What are the magnitude and direction of the learning effect sizes of using representations in Pre-K through fifth-grade mathematics when compared to traditional teaching methods?
2. What are the possible moderators that affect students' achievement and what classroom settings produce the most optimal learning effect sizes when representations are used?
3. What are the features of the most effective representations in mathematics suitable for Pre-K through Grade 5 levels?

It is hoped that the answers to these questions will advance the knowledge of using representations and assist math curriculum policymakers to design effective learning materials.

Data Collection Procedure

This meta-analysis sought to encompass 12 years of global research on using representations in Pre-K through fifth-grade mathematics, with student groups ranging in age from 3 to 12, in both public and private schools, with a minimum sample size of 15 participants. In the process of collecting the applicable research studies, ERIC (Ebsco), Educational Full Text (Wilson), Professional Development Collection, and ProQuest Educational Journals, as well as Science Direct, Google Scholar, and other resources available through a university library, were used to identify relevant studies published between January 1, 2000, and December 31, 2012. While extracting the relevant literature, the researchers used the following key terms: *graphical representations, mathematics education, primary, students, and experimental research*. In order to broaden the search, the terms *graphics, visualization*, and *problem-solving* were also used. Such defined criteria returned 131 papers, out of which 13 satisfied the conditions for meta-analysis (13 effect sizes). Several studies, although providing valuable findings, were rejected due their qualitative type (e.g., Castle & Needham, 2007) or due to their focus on comparing the effects of using representations that did not contain control groups (e.g., Coquin-Viennot & Moreau, 2003).

Coding Study Features
The main construct under investigation was the learning effect of using representations in Pre-K through fifth-grade mathematics classes. While some of the characteristics, for example, year of study conduct, locale, or type of research design, were extracted to support the study reliability, others, like grade level or intervention type, were extracted to seek possible moderators. Following are the descriptions of these features that were further aggregated to apply a subgroup moderator analysis.

Grade. This variable described the grade level of the group under investigation and referred to groups ranging from kindergarten to Grade 5.

Descriptive parameters. Descriptive parameters encompassed the locale where the studies were conducted, the date of publication, and the sample size representing the total number of subjects under investigation in experimental and control groups.

Publication bias. All studies included in this synthesis were peer-reviewed and published as journal articles; thus, no additional category for publication was created.

Group assignment. This categorization refers to the mode that was used to select and assign research participants to treatment and controlled groups. Two main groups were identified: (a) randomized, where the participants were randomly selected and assigned to the treatment and control group; and (b) quasi-experimental, where the participants were assigned by administrator selection. This categorization is aligned with Shadish, Cook and Campbell's (2002) definitions of group assignment.

Type of research designs used in the meta-analysis. Only pretest-posttest experimental studies with control groups were synthesized.

Intervention. The intervention (treatment approach) was classified into four categories reflecting the type of representations used in Pre-K through fifth-grade mathematics as defined by Swing, Stoiber, and Peterson (1988) and Xin and Jitendra (1999): pictorial (e.g., diagramming); concrete (e.g., manipulatives); mapping instruction (e.g., schemata based); and other (e.g., storytelling, keywords).

Output assessment. This variable described the assessment instrument and indicated whether the assessment was developed by the researcher or was standardized.

Data Analysis

The following analysis is organized deductively. It begins by describing the general study characteristics, moves to discuss the mean effect size, and concludes by presenting subgroup moderator computations. Such established sequence follows the order of the study research questions.

General Study Characteristics
The summary of the study characteristics extracted from the pool of experimental pretest-posttest studies is presented in Table 1.

Table 1. Tabularization of experimental pretest-posttest study features

Authors	Date	Locale	RD	SS	GL	IRT
Alibali, Phillips, & Fischer	2009	USA	QE	91	3rd & 4th	Pictorial
Van Oers	2010	Netherlands	QE	239	4th	Pictorial
Poland, Van Oers, &Terwel	2009	Netherlands	QE	54	2nd	Schemata based
Xin, Zhang, Park, Tom, Whipple, & Si	2011	USA	QE	27	4th	Schemata based
Booth & Siegler	2008	USA	R	52	1st	Pictorial
Csikos, Szitányi, &Kelemen	2012	Hungary	QE	244	3rd	Pictorial
Gamo, Sander, &Richard	2010	France	QE	261	4th & 5th	Schemata based
Terwel, Van Oers, Van Dijk, &Van den Eeden	2009	Netherlands	R	238	5th	Pictorial
Casey, Erkut,Ceder, & Young	2008	USA	QE	76	Pre-K	Other (storytelling)
Jitendra, Griffin, Haria, Leh, Adams, & Kaduvettoor	2007	USA	QE	88	3rd	Schemata based
Fuchs, Fuchs, Finelli, Courey, & Hamlett	2004	USA	R	436	3rd	Schemata based
Saxe, Taylor, McIntosh, & Gearhart	2005	USA	QE	84	4th & 5th	Pictorial
Fujimura	2001	Japan	R	51	4th	Concrete

Note. SS = sample size, GL = grade level, RD = research design, QE = quasi-experimental, R = randomized, IRT = intervention representation type.

The data revealed that there is substantial diversity in the representations used in elementary mathematics classes, ranging from schemas supporting problem-solving to storytelling supporting operations on fractions. The majority of the studies (nine, or 69%) were quasi-experimental, while four (31%) were randomized. Regarding grade, a dominating group of six studies was represented by fourth grade. Because problem-solving dominates the math learning objectives in K-12 math education, how representations help students improve their problem-solving techniques emerged as a possible moderator of the study. Considering the type of intervention, pictorial (six studies, or 46%) and schemata based (five studies, or 38%) dominated the pool.

The Mean Effect Size and its Significance

The quantitative inferential analysis in the form of a meta-analysis was performed on pretest-posttest experimental studies. The outcome variable, defined as the overall effect size of using representations in mathematics teaching was sought in this meta-analysis. Student achievement scores were further expressed as effect size computed using mean posttest scores of experimental and control groups and coupled standard deviation using Hedge's (1992) formula. For the meta-analytic methods to be applied, the responses for the experimental studies were standardized, and the accuracy of the effect sizes was then improved by applying Hedge's (1992) formula:

$$g = \frac{\bar{x}_1 - \bar{x}_2}{s*}$$

In this formula; $\bar{x}_1$ represents the posttest mean score of the treatment group, $\bar{x}_2$ represents the posttest mean score of the control group, and s^* represents pooled standard deviation. This process allowed the elimination of the sampling bias (Lipsey & Wilson, 2001).

The overall weighted mean effect size for the 13 primary studies (13 effect sizes) was reported to have a magnitude of 0.53 (SE = 0.05) and a positive direction. A 95% confidence interval around the overall mean – C_{lower} = 0.42 and C_{upper} = 0.63 – indicated a nonzero population effect and its relative precision (Hunter & Schmidt, 1990). According to Lipsey and Wilson (2001), an effect of 0.53 is of medium size. Herein, the overall effect's magnitude along with its positive direction indicated that the score of an average student in the experimental groups was 0.53 of a standard deviation above the score of an average student in the control groups. By incorporating the U_3 Effect Size Matrix (Cooper, 2010), the average pupil who was taught mathematics structures using representations scored higher on unit tests than 70% of students who were taught by traditional methods. Thus, it can be deduced that using representations in the teaching of mathematics, as a medium supporting instruction, has a profound impact on students' math concept understanding when compared to conventional methods of teaching. Therefore, contextualizing math ideas and letting students embed math operations in contexts meaningful to them has a positive effect on storing the ideas in their long-term memory. Table 2 provides a summary of the individual effect sizes of the meta-analyzed studies along with their confidence intervals and qualitative findings.

Table 2. Effect sizes of using representations in Pre-K through Grade 5

Study (First Author)	ES	SE	95% CI Lower	95% CI Upper	Research Findings	Source of Assessment
Alibali (2009)	0.92	0.22	0.19	1.05	The strategy of representing the process of equalizing equations improved problem representation techniques.	Researcher designed.
Van Oers (2010)	0.23	0.13	0.36	0.89	Children improved fraction understanding when they were allowed to construct own representations guided by the teacher.	Researcher designed.
Poland (2009)	1.22	0.29	0.04	1.23	Introducing dynamic schematizing improved understanding of the concept of the process during problem-solving.	Researcher-created schematizing test.
Xin (2011)	0.60	0.39	-0.19	1.44	Conceptual representations helped students learn the process of problem solving.	Used textbook items adopted by the districts; Cronbach 's alpha = 0.70.
Booth (2008)	0.20	0.28	0.05	1.19	Providing accurate visual representations of the magnitudes of addends and sums increased children's computational skills.	Wide Range Achievement Test–Expanded (WRAT–Expanded).
Csikos (2012)	0.62	0.13	0.36	0.88	Presenting word problems with different types of visualization (e.g., arrows) improved techniques of problem solving.	Test items adopted from National Core Curriculum; Cronbach's alpha = 0.83.
Gamo (2010)	0.61	0.14	0.34	0.91	Mapping data into graphical representations helped students with problems involving fractions.	Researcher designed.

Terwel (2009)	0.41	0.13	0.36	0.88	Having students learn to design representations helped them bring more model-based knowledge to the structure of mathematics problems.	Researcher developed criteria; Cronbach's alpha = 0.76.
Casey (2008)	2.00	0.31	0.38	2.63	Representing geometry concepts in a story context improved math knowledge retention.	Used Kaufman-Assessment Battery for Children (K-ABC; Kaufman & Kaufman, 1983).
Jitendra (2007)	1.36	0.22	-0.12	1.07	Addition and subtraction: used graphics to support multiple representations.	Used Pennsylvania System of School Assessment math test.
Fuchs (2004)	0.22	0.19	0.26	0.99	The applied schema for problem-solving improved students' algorithmic outcomes.	Researcher-developed.
Saxe (2005)	0.33	0.22	0.18	1.07	Percent: represented fractions with standard part-to-whole representations.	Researcher-developed.
Fujimura (2001)	0.71	0.29	0.05	1.20	Highlighting the idea of physical units in setting the proportions improved students' conceptual understanding.	Researcher developed; interrater agreement 97% (N = 76).

Note. ES = effect size, SE = standard error.

Calculated confidence intervals (CIs) for each effect size revealed that 11 of the effect sizes fell within 95% confidence intervals. The researcher used Statistical Package for the Social Sciences (SPSS) software to visualize the position of the effect sizes as well as the confidence intervals for each study around the computed overall mean of the pool of studies. Some means were revealed to be outside of the area of the funnel graph (see Figure 2).

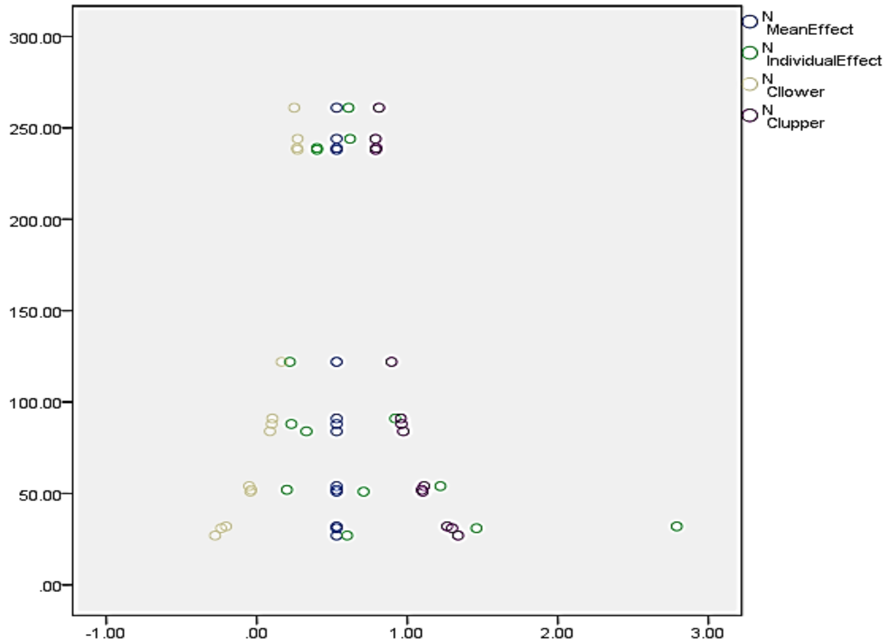


Figure 2. Funnel graph for the pretest-posttest experimental studies.

The individual effect sizes of some of the studies showed to be outside of the confidence intervals indicating a lack of homogeneity of distributions within the study pool. This was also depicted by the significant *p*-value ($p < 0.001$). As the purpose of a meta-analytic study is to compute effect size (Willson, 1983), the lack of homogeneity does not undermine the validity of the calculated mean effect; rather, it explicates the characteristics of the studies, indicating that some of them originated from different distributions.

The highest learning effect size (ES = 2.00) was generated in a study conducted with kindergarten pupils who were exploring the creation of verbal representations of geometry concepts (Casey, Erkut, Ceder, & Young, 2008). This study revealed that immersing math concepts in an environment that students can relate to their experiences and fantasies and letting students explore the links makes the math concepts tangible and results in them being easily stored in students' long-term memories. Another study with a high effect size (ES = 1.22), conducted by Poland, Van Oers and Terwel (2009), investigated the impact of dynamic representations on kindergarten students' math achievement. Dynamic representations provided more opportunities for having the learners explore their structures, thus generating a higher engagement factor and consequently higher learning effects. A positive learning effect of students' explorations was also advocated by Lesh and Harel (2003), who concluded that such situated learning enhances the processes of mathematical modeling that can play a vital role in developing students' scientific curiosity and they are problem-solving beyond the primary school level.

Analysis of Moderator Effects

The process of a further synthesis of the studies' features allowed for identifying the following moderators: treatment length, mode of introducing the representations, grade level, and content standards. The moderators were further disaggregated by their levels. Where applicable, the levels within the moderators were contrasted, and inferences on differences were made. The following criteria were applied to disaggregate the moderators.

Treatment length. The treatment length followed a partition established by Xin and Jitendra (1999): short – less than one week; intermediate – between 1 week and one month; and long - more than one month.

Mode of representation induction in the lesson cycle. This moderator followed operational roles of representations and contained two levels: concept introduction and problem-solving.

Grade level. The large range of grades was distributed into two levels according to standard classification (NCTM, 2000). The lower group level encompassed all students from Pre-K to Grade 3, and the upper level included Grades 4 and 5.

Content standards. This moderator reflected general standards examined in the studies that were clustered into the following: number and operations, proportions, and geometry. A summary of the weighted effect sizes is presented in Table 3.

Table 3. Summary of subgroups' moderator effect sizes

Moderator and Levels	N	ES	SE	95 % CI	
				Lower	Upper
Grade Level					
Lower: Pre-K through 3	6	0.60	0.08	0.45	0.76
Upper: 4-5	7	0.47	0.07	0.33	0.60
Representation Type					
Pictorial	6	0.45	0.06	0.32	0.57
Schemata based	5	0.49	0.09	0.31	0.67
Concrete	1	0.71	0.29	0.05	1.20
Other	1	2.00	0.31	1.38	2.63
Treatment Length					
Short	5	0.46	0.07	0.31	0.61
Intermediate	4	0.53	0.10	0.33	0.72
Long	4	0.60	0.10	0.40	0.80
Content Standard					
Numbers and operations	10	0.45	0.06	0.34	0.56
Geometry	2	1.61	0.22	0.17	0.24
Ratio and proportions	1	0.71	0.29	0.05	0.20
Mode of Induction					
Concept Introduction	7	0.68	0.07	0.54	0.82
Problem-solving	6	0.49	0.08	0.34	0.64

Note. N = number of participants, ES = effect size, SE = standard error.

The computing of the mean subgroup effect sizes provided a basis for answering more detailed research questions. When compared by grade level, the effect of using representations was higher in Pre-K through Grade 3 than in Grades 4-5. This conclusion might be supported by the fact that as students' progress with learning math concepts, they learn more abstract semantics that might be difficult to embody in representations, for instance, the idea of fraction division. Students can follow the initial and final stage of the process. However, the diversity of the methods of dividing that is embodied by the syntax of division along with the various representations of rational expressions might not be entirely comprehended and thus it needs more clarity.

When mediated by the type of representation, *concrete* and *others* produced the highest effect sizes; yet, their significance could not be fully apprehended because each subgroup was represented by a single primary study. When pictorial representations (ES = 0.45) and schemata-based representations (ES = 0.49) were contrasted, schemata representations showed a higher impact on student learning, which supports the findings of other scholastic research (e.g., Jitendra et al., 2007; Terwel, Van Oers, Van Dijk, &Van den Eeden, 2009; Xin et al., 2011). Overall, schemata-based representations and their applications emerged as the main type of representations supporting problem-solving. According to Owen and Sweller (1985), a schema is a general cognitive structure that allows the learner to categorize the problem and then apply specific tools to solve it. A moderate effect size (ES = 0.49) indicates that this learning strategy helps students understand underlying mathematical ideas in given word problems and solve them. Hiebert and Carpenter (1992) posited that while developing the schemas, students activate a complex network of concepts stored in their long-term memory. Furthermore, the networks constitute the model that will be called an internal representation of the domain embodied by an external representation. As Xin et al. (2011) suggested, instead of telling students, for instance, the numerical magnitude of the unit rate, e.g., ten apples per basket, demonstrating students a real concrete representation along with its symbolic mathematical model will better support the conceptual understanding and its mathematical embodiment.

How should representations be furnished to effectively develop the conceptual networks? Learners can be provided with the representations, or the representations can be derived by the learners under the teacher's guidance. It is inferred from this study that providing students with opportunities to derive the representations deems to be a more effective teaching strategy because it allows the learners to retain the concepts longer and apply them in new situations more frequently.

An interesting direct variation was observed when the effect sizes were contrasted with treatment lengths. It became apparent, from examining this relation, that the longer the treatment, the higher the effect size (ES = 0.46 for short treatments, ES = 0.53 for intermediate, and ES = 0.60 for long). This result provides support for applying representations in classes daily.

Concerning content standards, geometry representations yielded a higher effect size (ES = 1.61). This result reflects the visual nature of content in this branch of mathematics, which by virtue is rooted in representations. The analysis of the concluding subgroup—mode of inducing in the lesson cycle—allowed contrasting the effect sizes of using representations to support conceptual understanding and problem-solving. It is apparent that representations are more effective with concept introduction (ES = 0.69) than problem solving (ES = 0.49). Thus, one could conclude that supporting concept introduction with representations builds a stronger network of impulses in students' long-term memory.

Summing all these findings led to the formulation of a classroom setting that would generate the highest learning effect sizes. It seems that using concrete representations to introduce geometry concepts in Pre-K through Grade 3 would yield the highest learning effects.

The research findings also allow for the formulation of recommendations for effective representations. Fujimura (2001) concluded that representations should share similar features as the target domain and must be manipulative so that children can explore and uncover embedded math structures by themselves. He further suggested that representations should be designed in a way that they develop children's creativity in constructing mathematical models. Casey et al. (2008) found that students retain mathematical knowledge if the knowledge is embedded in a story context. Developing mathematical knowledge through sequenced mathematics problems related to the storyline is also suggested by the researcher of the meta-analysis. Booth and Siegler (2008) highlighted accuracy and transparency of representations as a significant factor affecting students' mathematical learning in early grades, whereas Poland et al. (2009) brought forth the idea of using dynamic representations to support the processes of arithmetic operations.

As mathematics seeks to develop students' concise, abstract thinking, the results of this synthesis show that it also needs to reflect on representations that students use in daily life and whose contents are adequate to their experiences. Presenting artificially created representations that do not adhere to students' experiences might disconnect mathematics concepts from the realm and rather support the notion that mathematics is an abstract subject.

General Conclusions

The findings of this study support the following hypothesis: representations help Pre-K through fifth-grade students learn and apply abstract math concepts, especially when such representations are applied to support new concept understanding and students' problem-solving skills. Certain limitations and recommendations for further research emerged from this study, as discussed below.

Threats to Research Validity

The primary parameter limiting the study findings was a lower-than-expected pool of primary studies that satisfied the conditions to be meta-analyzed. The validity of the study computations was supported by double research data coding at the initial and concluding stages of the study process. Any potential discrepancies were resolved. Although strictly specified, the literature search was undertaken with broader conceptual definitions in mind that allowed for, as suggested by Cooper (2010), adjustment of the definitions and strengthening of the literature relevance. Thus, as the initial literature search revealed that representations in Pre-K through Grade 5 are often used to support problem-solving, the term *problem solving* was then used to locate more studies.

Schemata as a Major Type of External Representations

Among different representations (see Table 1) schemata-based representation and their applications emerged as the most commonly used to support problem-solving. According to Owen and Sweller (1985), a schema is a general cognitive structure that allows the problem solver to categorize the problem and then apply certain tools to solve it. A moderate effect size (ES = 0.49) indicates that this learning strategy helps students understand underlying math ideas in given word problems. Hiebert and Carpenter (1992) posited that in the process of developing the schemata, students' thinking blends a complex network of concepts in one coherent picture. Furthermore, the networks constitute a mental model that will be called an internal representation of the domain imaged by an external representation. The conceptual networks can be developed either by representations provided by the teacher or by representations derived by the learners under a teachers' guidance.

Is having students use schemata sufficient to have them learn the holistic picture of the meaning of this mathematical representation? Several researchers concluded that once children are exposed to certain representations – for instance, schematic representations to solve problems – they retain those methods and apply the schemas regardless of age (Coquin-Viennot & Moreau, 2003). Some scholars noted (e.g., Castle & Needham, 2007), this idea cannot be overemphasized; children also need some *working space* to analyze problems and devise their ways to solve the problems with the *support* of provided schemata. Thus schemata should be perceived as suggestions for mathematization of certain patterns not as fixed formulas to use. It seems that more research should focus on having students recognize the type of scientific underpinning of the problem that students should apply to determine the principles embedded in a given word problem.

Hiebert and Carpenter (1992) proposed four semantic categories (schemata) for arithmetical operations that are: *change, combine, compare,* and *equalize.* Using these schemata to model *story problems* allows certain flexibility, for example, in some cases can be perceived as *compare*, or *compare* can suffice to *combine*. Emphasizing the schemata to reach the final solution reduces learners' opportunities to explore and be immersed in the process of analyzing the problem structure. There can also be cases when two or more schemata can be used in successions. For example, in order to *compare*, students might need to *combine* similar elements first. Thus students should be allowed to exhibit flexibility in applying the schemata and interpret them. However, that the primary meaning of each should be consistently executed. To illustrate that consider the following problem discussed by Jitendra et al., (2007): *Music Mania sold 56 CDs last week. It sold 29 fewer CDs last week than this week. How many CDs did it sell this week?* This problem was intended to support the idea of *compare.* There is merit to use the schema of *compare* in this problem, but is the schema *compare* the most representative to mathematize the process of selling the CDs? The problem mentioned two events happening at two different time instants referring to similar objects, can then the learner be directed to considering rather finding the difference? Thus would the schemata of *change* better describe the process and elicit its solution? It seems that referring students to *compare* gears their thinking toward the output of the problem thus finding the final product, not toward the principal process, *the change* that supported the process of reaching the output. By directing students' attention to the problem output, the phase of problem analysis is reduced. Referring to the problem context and considering the definition of change as *Change = Final value – Initial value,* and solving for *Change,* one will receive *Change = This week sells – Last week sells.* Substituting the given values results in *29 = This week sells – 56*, that leads further to *This week sells = 56 + 29* which leads to the conclusion that *Music Mania* sold 85 CDs. With the implementation of *change,* the representation involved negative numbers that perhaps were not intended in Jitendra's study. Thus to further discuss the applicability of this problem to Grade 3 math curriculum, the problem would have to be redesigned, however providing students with the flexibility of exercising the underlying *process* that is missed is worth further research. Zooming further change in quantity values is concluded by subtracting the initial value from the final value: *Change = Ending – Beginning*. This standard definition of change is applied not only in mathematics to calculate, for example, instantaneous or average rate of change (e.g. Stewart, 2006) but also in sciences, especially in physics where the concept of change is often used to calculate change of temperature, or object's displacement (e.g., Giancoli, 2005).

One might be interested in learning how the schemata of *change* are induced in the literature. The idea of using *change* was introduced by Marshall (1995) see Figure 3 and was modeled by the following problem; *Jane had 4 video games. Then her mother gave her 3 video games for her birthday. Jane now has 7 video games.*

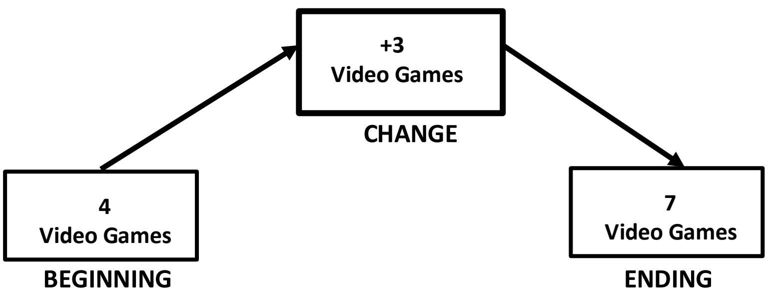


Figure 3**.** Representation of *change* inspired by Marshall (1985).

Change in quantities values is concluded by subtracting the initial value from the final value: *Change = Ending – Beginning*. This standard definition of change is applied not only in mathematics to calculate, for example, instantaneous or average rate of change (e.g. Stewart,

2006) but also in sciences, especially in physics where the concept of change is often used to calculate change of temperature, or object's displacement (e.g., Giancoli, 2005). This equation can be rearranged to reflect Marshal's idea *Beginning + Change = Ending.* However, the rearranged form is not aligned with the fundamental principle of the *process of change* that seems to be the core idea of the problem. If the schema of *change* were to be used, then perhaps the diagram could have been redesigned to reflect on the difference in the quantity amount that represents the *change*.

These two examples were brought up to signify a need for verifying interdisciplinary consistency of the interpretations of the fundamental concepts that are meant to support problem-solving in K-5 mathematics. It is understood that the equations symbolizing the schemata can be rearranged and executed with a dose of flexibility. What stages were being used would depend on individual perception, yet general foundations for problem analysis would perhaps require more consistency. Perhaps establishing fewer schemata and letting students manipulate on them to reflect the process of a given problem could be an alternative avenue to pursue? The mathematical operations behind calculations of *change, combine, compare,* and *equalize* are very fundamental in sciences, and it seems that understanding their core meanings might have a profound impact on students success on problem-solving not only at an elementary but also at a high school level and beyond.

Looking Ahead: Linking the Representations

Cheng (1999) proposed four learning stages that can help students in developing conceptual understanding through using representations: domain, external representation, concept, and the internal network of concepts. While moving from one stage to another to reach the internal network, the learner is immersed in four processes: observation, modeling, acquisition, and integration. Except the study conducted by Rittle-Johnson, Siegler, & Alibali (2001) and Terwel et al. (2009), the majority of the gathered research did not explicate on these processes, focusing instead on applying fixed models without discussing their possible modifications.

It seems that possessing the right representation does not suffice for an understanding. To confirm an understanding, one needs to be able to put this representation through its paces, explaining and predicting novel cases. Thus, to have an understanding of a representation is to be in a state of readiness, taking the representation as a point of departure in the solution process, not as an unquestionable formula a representation. Terwel et al. (2009) proved that having students explore and modify given representations produced the highest effect size. This can be supported by the effects of induced math modeling phases that allowed the students to link representations with the constraints of real scenarios (Sokolowski, 2018).

Applying representations often creates exploratory learning environments (English & Watters, 2005) that consequently can be guided by inductive or deductive inquiry processes. Thus, other themes worthy of a further investigation emerged; should the use of representations to be organized inductively, as suggested by Nunokawa (2005)? How do elementary school students perceive these two major scientific inquiries? Are these inquiries rooted in virtues of mathematical representations, or content-domain? Having students develop principles of representations by identifying commonalities due to applications and use such representations to model other contexts beyond the boundary of a math classroom would be an interesting pursuit for future studies.

Bridging Representations Used at Elementary and High School Levels

There are other questions can be generated from this study, for example; how does the use of representations evolve as students' progress with their mathematics classes, especially schemata-based that dominate problem-solving in Pre-K through Grade 5. Fuchs, L., Fuchs, D., Finelli, Courey, & Hamlet (2004) suggested using schemata more extensively for problem-solving also at the high school level, especially targeting students with learning disabilities. Having high school students derive processes of transitioning from, for example, proportion to a linear or rational function or from percent rate to an exponential function seem like valuable topics to explore.

Another conclusion calls for extending the idea of using schemata to sciences and other subjects in a consistent manner that will carry out their general principles. This transition would help broaden the meanings and consequently built a stronger image of these representations in students' long-term memories. Do students experience applications of similar representations in their science classes? Should these main avenues of knowledge acquisition depend on the nature of the representation (schemata or pictorial) or their general purpose? Further research in these areas is needed, and it is believed that this paper provides some prompts for such actions.

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Corresponding author: Andrzej Sokolowski
Contact email: Andrzej.Sokolowski@lonestar.edu